Structures and lower bounds for binary covering arrays

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Notations

- $B_q = \{0, 1, \ldots, q-1\}$.
- For $u = (u_1, u_2, \ldots, u_n) \in B_q^n$,
  - $\mathrm{supp}(u) = \{i \mid u_i \neq 0\}$.
  - $wt(u) = |\mathrm{supp}(u)|$.
- $[n] = \{1, 2, \ldots, n\}$.
- For $C = (c_{ij})$ over B_q, c^i is the i-th column of C.

Definition

An $m \times n$ matrix C over B_q is called a t-covering array (or, a covering array of size m, strength t, degree n, and order q) if, in any t columns of C, all q^t possible q-ary t-vectors occur at least once. We denote such an array by $CA(m; t, n, q)$.

Example

The following matrix is a 2-covering array over B_2.

$$
\begin{matrix}
1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{matrix}
$$

Definition

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Applications

- circuit testing,
- intersecting codes,
- data compression.

- The main problem is to optimize one of the parameters m and n for given value of the other:
  (a) find the minimum size $CAN(t, n, q)$ of a t-covering array of given degree n over B_q;
  (b) find the maximum degree $\overline{CAN}(t, m, q)$ of a t-covering array of given size m over B_q.
- $q^t \leq CAN(t, n, q) \leq q^n$.
- Rènyi (for m even), and independently Katona, and Kleitman and Spencer (for all m) showed that $\overline{CAN}(2, m, 2) = \binom{m-1}{\lfloor \frac{m}{2} \rfloor - 1}$.
- Johnson and Entringer showed that $CAN(n-2, n, 2) = \lfloor \frac{2^n}{3} \rfloor$.
- Colbourn et al. give all the known upper and lower bounds for covering arrays up to degree 10, order 8 and all possible strengths, but their classification results are much more limited.

Theorem

(*G. Roux 1987*)

$$CAN(t+1, n+1, q) \geq qCAN(t, n, q),$$
$$CAN(3, 2n, 2) \leq CAN(3, n, 2) + CAN(2, n, 2).$$

Example

The following matrix is a 2-covering array over B_2.

$$\begin{matrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

Example

$$\begin{array}{cccc} \color{blue}{1} & \color{blue}{0} & \color{blue}{0} & \color{blue}{0} \\ \color{blue}{1} & \color{blue}{1} & \color{blue}{1} & \color{blue}{1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}$$

Permutation of the rows

$$\begin{array}{cccc} \color{blue}{1} & \color{blue}{1} & 1 & 1 \\ \color{blue}{0} & \color{blue}{1} & 0 & 0 \\ \color{blue}{1} & \color{blue}{0} & 0 & 0 \\ \color{blue}{0} & \color{blue}{0} & 1 & 0 \\ \color{blue}{0} & \color{blue}{0} & 0 & 1 \end{array}$$

Permutation of the columns

$$\begin{array}{cccc} \color{blue}{0} & 1 & 1 & 1 \\ \color{blue}{0} & 0 & 0 & 0 \\ \color{blue}{1} & 1 & 0 & 0 \\ \color{blue}{1} & 0 & 1 & 0 \\ \color{blue}{1} & 0 & 0 & 1 \end{array}$$

Permutation of the values of any column

Definition

Two covering arrays C and C' are equivalent if one can be transformed into the other by a series of operations of the following types:
(a) permutation of the rows;
(b) permutation of the columns;
(c) permutation of the values of any column.

- Katona proved that maximal binary covering arrays of strength 2 are uniquely determined up to equivalence.
- Johnson and Entringer showed that $\lfloor \frac{2^n}{3} \rfloor \times n$ binary covering arrays of strength $n-2$ are uniquely determined up to equivalence.

Goals

- Classify the structures of some optimal binary 2-covering arrays.
- Improve the lower bound of Roux on $CAN(t, n, q)$ when $t = 3, q = 2$.

- For $u \in B_2^n$, $\overline{u} = (\overline{u}_1, \ldots, \overline{u}_n)$ where

$$\overline{u}_i = \begin{cases} 1, & \text{if } u_i = 0; \\ 0, & \text{if } u_i = 1. \end{cases}$$

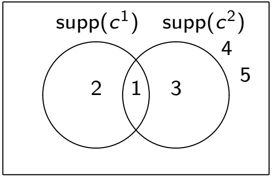
- $u \in B_2^n \Leftrightarrow \mathrm{supp}(u) \subseteq [n]$
- The following statements are equivalent.
  - C is a binary t-covering array.
  - $\cap_{k=1}^{t} X_{i_k} \neq \emptyset$ for $\{i_1, \ldots, i_t\} \subseteq [n]$, where X_k is either $\mathrm{supp}(c^k)$ or $\mathrm{supp}(\overline{c^k})$.

$$C = \begin{array}{c|cccc} & c^1 & c^2 & c^3 & c^4 \\ 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 & 0 \\ 4 & 0 & 0 & 1 & 0 \\ 5 & 0 & 0 & 0 & 1 \end{array}$$

$$\mathrm{supp}(c^1) = \{1,2\}$$
$$\mathrm{supp}(c^2) = \{1,3\}$$
$$\mathrm{supp}(c^3) = \{1,4\}$$
$$\mathrm{supp}(c^4) = \{1,5\}$$

$$C = \begin{array}{c|cccc} & c^1 & c^2 & c^3 & c^4 \\ \hline 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 & 0 \\ 4 & 0 & 0 & 1 & 0 \\ 5 & 0 & 0 & 0 & 1 \end{array}$$

$$\begin{aligned} \operatorname{supp}(c^1) &= \{1,2\} \\ \operatorname{supp}(c^2) &= \{1,3\} \\ \operatorname{supp}(c^3) &= \{1,4\} \\ \operatorname{supp}(c^4) &= \{1,5\} \end{aligned}$$



Definition

The standard maximal binary 2-covering array C of size m is an $m \times \binom{m-1}{\lfloor \frac{m}{2} \rfloor - 1}$ matrix with
(1) the first row of C is all 1 row,
(2) the columns of the remaining matrix is the family of all vectors of $(\lfloor \frac{m}{2} \rfloor - 1)$ 1's and $(m - \lfloor \frac{m}{2} \rfloor)$ 0's.

Example

$$\begin{matrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{matrix}$$

Theorem

(*E. W. Hall 1935*)
Suppose we have a bipartite graph G with two vertex sets V_1 and V_2. Suppose that

$$|\Gamma(S)| \geq |S| \qquad \textit{for every } S \subset V_1.$$

Then G contains a complete matching.

Lemma

Let C be a 2-covering array of size m and degree n with $wt(c^i) \leq \lfloor \frac{m}{2} \rfloor$ for all $1 \leq i \leq n$. Put $s = min_{1 \leq i \leq n} wt(c^i)$. For any integer s' satisfying $s < s' \leq \lfloor \frac{m}{2} \rfloor$, there is a 2-covering array C' of size m and degree n with $s' \leq wt(c'^i) \leq \lfloor \frac{m}{2} \rfloor$ such that $supp(c^i) \subseteq supp(c'^i)$ for all $i \in [n]$.

Corollary

Let C be a 2-covering array of size m and degree n with $wt(c^i) \leq \lfloor \frac{m}{2} \rfloor$ for all $i \in [n]$ and $wt(c^j) < \lfloor \frac{m}{2} \rfloor$. Then there is a 2-covering array C' of size m and degree n with $wt(c'^j) = \lfloor \frac{m}{2} \rfloor - 1$ and $wt(c'^i) = \lfloor \frac{m}{2} \rfloor$ for all $i \in [n]$ and $i \neq j$ such that $supp(c^i) \subseteq supp(c'^i)$ for all $i \in [n]$.

Theorem

(A. J. W. Hilton, E. C. Milner 1967)
Let $2 \leq k \leq \frac{m}{2}$. *Let* C *be a binary 2-covering array of size* m *such that* $wt(c^i) \leq k$ *for any column of* C *and* $\bigcap_{1 \leq i \leq n} supp(c^i) = \emptyset$. *Then*

$$n \leq d = 1 + \binom{m-1}{k-1} - \binom{m-k-1}{k-1}.$$

There is strict inequality if $wt(c^i) < k$ *for some* $i \in [n]$.

Theorem

Let $m \geq 4$, $k = \lfloor \frac{m}{2} \rfloor$, *and* $\binom{m-1}{k-1} + m - 3k + 1 \leq n \leq \binom{m-1}{k-1}$.
If an $m \times n$ *matrix* C *over* B_2 *is a 2-covering array, then* C *is equivalent to the matrix made from deleting columns of standard binary 2-covering.*

Corollary

Every maximal binary 2-covering arrays is equivalent to the standard maximal 2-covering array.

Corollary

If $m \geq 6$ and $n = \binom{m-1}{\lfloor \frac{m}{2} \rfloor - 1} - 1$, then every $m \times n$ binary 2-covering array C is made from deleting a column of the standard maximal 2-covering array.

- $10 \times 5, 12 \times 11$ binary optimal 3-covering and 24×12 binary optimal 4-covering arrays are unique.
- There is no 48×13 binary 5-covering array.

Theorem

If $m \geq 7$, $k = \lfloor \frac{m}{2} \rfloor$, *and* $\binom{m-1}{k-1} + m - 3k + 1 \leq n \leq \binom{m-1}{k-1}$, *then*

$$CAN(3, n+1, 2) \geq \begin{cases} 2CAN(2, n, 2) + 1 & \textit{if } m \textit{ is odd} \\ 2CAN(2, n, 2) + 2 & \textit{if } m \textit{ is even} \end{cases}$$

n	6	7	8	9	10
$CA(6;2,n,2)$	4	3	1	1	1

Table 1 : The number of covering arrays $CA(6;2,n,2)$.

n	32	33	34	35
$CA(8;2,n,2)$	5	2	1	1

Table 2 : The number of covering arrays $CA(8;2,n,2)$.

n	Lower Bound of Roux	Lower Bound of C.,Kim, Oh	Upper Bound
4	8	8	8
5	10	10	10
6-11	12	12	12
12	14	14	15
13-16	14	15	16-17
17-31	16	16	18-24
32-36	16	18	24
37-53	18	18	24-29
54-57	18	19	29-31
58-121	20	20	31-33
122-127	20	22	33
...	...	...	...
1710-1717	28	30	66-67
...	...	...	...
6428-6436	32	34	74

Table 3 : Tables of $CAN(3, n, 2)$.

Thank you for your attention!